Algebraic Geometry Mid-Semestral Examination, 2009, B.Math 3rd Yr/M.Math 2nd Yr

Each question carries 10 marks, and books and notes maybe used. Results proved in class may be used without proof, but results of exercises need to be proved in full. In the sequel, you may assume the field k to be algebraically closed unless stated otherwise..

1. (i): Let $\mathfrak{a}$ be an ideal in a Noetherian ring. Show that $\left(\sqrt{\mathfrak{a}}\right)^n \subset \mathfrak{a}$ for some $n \geq 1$.

   (ii): Let A be a Noetherian ring, and $f : A \to B$ be a ring homomorphism. Make B into an A-module via f, viz. define $a * b := f(a)b$. Suppose that B is a finitely generated A-module with this module structure. Then show that B is a Noetherian ring.

2. (i): Show that the ring extension $k[X] \subset k[[X]]$ is not an integral extension.

   (ii): Show that the graph of the exponential function $f(x) = e^x$ is Zariski dense in $\mathbb{R}^2$.

3. (i): Let k be a field (not necessarily algebriacally closed) and K be its algebraic closure. Show that there is a $1-1$ correspondence between k-algebra homomorphisms $f : k[X_1, .., X_n] \to K$ and points $(a_1, .., a_n) \in \mathbb{A}^n(L)$ where L is a finite algebraic extension of k (depending on f).

   (ii): Show that the ring $A := k[X,Y]/\mathfrak{a}$ where $\mathfrak{a} = \langle Y^2 - X^3 \rangle$ is not integrally closed in its quotient field.

4. (i): Decompose the algebraic set $V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subset \mathbb{A}^2(k)$ into its irreducible components.

   (ii): Let $X = \cup_{i=1}^m X_i$ be the irredundant decomposition of an affine algebraic set X into its irreducible components, and let $x \in X_i \setminus (\cup_{j \neq i} X_j)$. Show that there exists a function $f \in \mathcal{O}(X)$ such that $f(x) \neq 0$ and $f_{|X_j} \equiv 0$.